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COLOR PARTITIONS AND GANDHI'S RECURRENCE RELATION

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ABSTRACT. We employ the Z-transform to solve a type of recurrence relation satisfied by $p_k(n)$, the number of k-colored partitions of n; the corresponding solution is in terms of the complete Bell polynomials and it is equivalent to a recent formula deduced by Alegri. Thus, $p_k(n)$ is a polynomial in k of degree n whose coefficients are determined by the compositions of n.

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1. INTRODUCTION

We know the Gandhi's recurrence relation [1, 2, 3, 4, 5, 6, 7, 8]:

$$n\, p_k(n) = k \sum_{j=1}^{n} \sigma(j)\, p_k(n-j), \tag{1}$$

where σ is the sum of divisors function [9, 10, 11, 12, 13] and $p_k(n)$ is the number of k-colored partitions of n [2, 3, 14, 15, 16].

Then it is natural to study recurrence relations with the structure of a Cauchy convolution [11]:

$$n\, f_k(n) = \sum_{j=1}^{n} h(j) f_k(n-j), \qquad k \geq 1, \;\; n \geq 0, \tag{2}$$

verifying the properties $f_k(0) = 1 \;\; \forall\; k$ and $h(0) = 0$.

In Sec. 2 we employ the Z- transform [17, 18, 19] to (2) to deduce the following solution:

$$f_k(n) = \frac{1}{n!} B_n\Big(0!h(1), 1!h(2), 2!h(3), ..., (n-1)!h(n)\Big), \tag{3}$$

in terms of the complete Bell polynomials [20, 21, 22, 23, 24, 25, 26, 27, 28]. In Sec. 3 we apply (3) to $p_k(n)$ to obtain a relation in harmony with a recent expression of Alegri [16]. From (1) it is immediate to see that $p_k(n)$ is a polynomial in k of degree n, which also is analyzed in Sec. 3.

2. Z-TRANSFORM APPLIED TO (2)

If $F(z)$ and $G(z)$ are the Z-transforms of the sequences $\{f_k(0), f_k(1), f_k(2), ...\}$ and $\{0, h(1), ...\}$ respectively:

$$F(z) = 1 + \frac{f_k(1)}{z} + \frac{f_k(2)}{z^2} + ..., \qquad H(z) = \frac{h(1)}{z} + \frac{h(2)}{z^2} + ..., \tag{4}$$

then (2) gives the differential equation:

$$-z\frac{d}{dz}F = H(z)F(z), \tag{5}$$

whose integration implies the solution:

$$Ln\ F = \frac{h(1)}{z} + \frac{h(2)}{2z^2} + \frac{h(3)}{3z^3} + ... \tag{6}$$

that is:

$$F(z) := \sum_{n=0}^{\infty} f_k(n)\frac{1}{z^n} = exp\Big(\sum_{j=1}^{\infty}\frac{h(j)}{j}\frac{1}{z^j}\Big) \tag{7}$$

On the other hand, we have the generating function of the complete Bell polynomials [26]:

$$\sum_{n=0}^{\infty}\frac{1}{n!}B_n\Big(x_1, x_2, ...x_n\Big)\frac{1}{z^n} = exp\Big(\sum_{j=1}^{\infty}\frac{x_j}{j!}\frac{1}{z^j}\Big), \tag{8}$$

whose comparison with (7) implies (3), q.e.d.

3. Application of (3) to $p_k(n)$

The result (3) gives the solution of recurrence relations with the structure (2), then its application to (1) allows obtain the following interesting explicit expression for color partitions:

$$p_k(n) = \frac{1}{n!}B_n\Big(k\cdot 0!\ \sigma(1), k\cdot 1!\ \sigma(2), k\cdot 2!\ \sigma(3), ..., k\cdot(n-1)!\ \sigma(n)\Big), \tag{9}$$

which it is equivalent to the recent result of Alegri [16]:

$$p_k(n) = \sum_{j=1}^{n}\frac{k^j}{j!}\sum_{(\omega_1,...,\omega_j)\in C_n}\frac{\sigma(\omega_1)...\sigma(\omega_j)}{\omega_1...\omega_j}, \tag{10}$$

in terms of compositions of n.

A composition of n is an ordered collection of positive integers whose sum is n. The set of compositions of n is denoted by C_n. There are eight compositions of 4:

$$C_4 = \Big\{(4), (3,1), (1,3), (2,2), (2,1,1), (1,2,1), (1,1,2), (1,1,1,1)\Big\}. \tag{11}$$

A partition of an integer n is a non-increasing sequence of natural numbers whose sum is n. The partitions of $n = 4$ are $4, 3{+}1, 2{+}2, 2{+}1{+}1, 1{+}1{+}1{+}1$, that is, $p(4) = 5$. A k-colored partition of n is a partition of n in which each part appears colored with one of the k available colors. For example, if $k = 2$, and the colors considered are black and red, the 2-color partitions of $n = 4$ are 4, 4, 3 + 1, 3 + 1, 3 + 1, 3 + 1, 2 + 2, 2 + 2, 2 + 2, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1, that is, $p_2(4) = 20$, this value results from (11) and (10) with $k = 2$ and $n = 4$. Besides, from (9):

$$p_2(4) = \frac{1}{24}B_4\Big(2\ \sigma(1), 2\ \sigma(2), 4\ \sigma(3), 12\ \sigma(4)\Big) = \frac{1}{24}B_4\Big(2, 6, 16, 84\Big), \tag{12}$$

where we can use the complete Bell polynomial:

$$B_4\Big(x_1, x_2, x_3, x_4\Big) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4, \tag{13}$$

to obtain again that $p_2(4) = 20$.

From (1), (9) and (10) it is clear that $p_k(n)$ is a polynomial in k of degree n, in fact, $p_k(0) = 1,\ k \geq 1$ [29, 30]:

$$p_k(1) = k, \quad p_k(2) = \frac{1}{2!}k(k+3), \quad p_k(3) = \frac{1}{3!}k(k+1)(k+8),$$

(14)

$$p_k(4) = \frac{1}{4!}k(k+1)(k+3)(k+14), \quad p_k(5) = \frac{1}{5!}k(k+3)(k+6)(k^2+21k+8), ...,$$

with the properties $p_1(n) = p(n)$ and:

$$p_{-1}(n) = a_n = \begin{cases} (-1)^m, & n = \frac{m}{2}(3m+1), \quad m=0, \pm 1, \pm 2, ... \\ 0, & otherwise \end{cases} \tag{15}$$

hence:

$$p_k(n) = a(n,n)k^n + a(n,n-1)k^{n-1} + ... + a(n,2)k^2 + a(n,1)k, \tag{16}$$

whose comparison with the relation (10) obtained by Alegri [16] allows give a formula for the coefficients of this polynomial in terms of the compositions of n and the corresponding values of the sum of divisors function:

$$a(n,j) = \frac{1}{j!} \sum_{(\omega_1,...,\omega_j)\in C_n} \frac{\sigma(\omega_1), ..., \sigma(\omega_j)}{\omega_1, ..., \omega_j}, \quad j = 1, 2, ..., n; \tag{17}$$

then, for example, with (11), (16) and (17) is easy to construct the polynomial shown in (14):

$$p_k(4) = \frac{1}{24}k^4 + \frac{3}{4}k^3 + \frac{59}{24}k^2 + \frac{7}{4}k. \tag{18}$$

4. Conclusions

We have shown that $p_k(n)$ admits an expression in terms of the complete Bell polynomials whose arguments are values of the sum of divisors function, in harmony with the corresponding formula obtained by Alegri [16] based on the compositions of n. We consider that an analysis like the one carried out here can be carried out for other arithmetic functions verifying a recurrence relation of type (2).

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